Elasticity

Economics 100A

Fall 2021

1 Elasticity

1.1 Elasticity Overview

Elasticity is a measure of responsiveness. More specifically, we measure the change in variable due to the change of another variable. For example, if the price of a good changes, how does the quantity demanded change? Since we know calculus, we can measure this quite easily. Assume that we have an independent and a dependent good. Our set up will begin as follows:

$$y = f(x) \tag{1}$$

If we change x, how does y change in response? Let's start with a very simple case. Say $x_1 = 5$ and decreases to $x_1 = 2$. Accordingly, y_1 starts at 4 but increases to 6. We can calculate the change in each variable as the ratio of the changes:

$$\frac{y_2 - y_1}{x_2 - x_2} = \frac{\Delta y}{\Delta x} \tag{2}$$

In the example above, we would write $\frac{6-2}{2-5}$ which is just $\frac{4}{-3}$. Therefore, we see that we have a negative relationship between the two variables.

As you may remember from your calculus classes, $\frac{\Delta y}{\Delta x}$ is just the slope of a function at point x. In other words, it is a derivative. So, we can see that initially, things are as simple as a derivative. However, imagine the following scenario: the price of a good increases by \$100 and quantity falls by 1000 units. Is this a large change? Well, for a car, the price change is not much. For a meal, however, the change is steep. We need a better way of finding the responsiveness of variables. Currently, we are using absolute change. Can you think of a better way of representing change?

If you thought about converting absolute change into percentage change, you are correct. Let us think back to our initial, simpler problem, where x_1 changes from 5 to 2 and y_1 increases from 4 to 6. To measure the percentage changes in each of these variables, we need to rewrite our fraction to be:

$$\epsilon = \frac{\dfrac{y_2 - y_1}{y_1}}{\dfrac{x_2 - x_1}{x_1}} = \frac{\%\Delta y}{\%\Delta x} \tag{3}$$

Now, this is what would be called a continuous equation for elasticity. This is because we are measuring the elasticity between two different points. But what if you are not given two different points? What if you need to solve for the elasticity at a discrete point?

If you are thinking about taking a derivative and converting it into a percent, you are right again. Let's write this out in Leibniz's notation.

$$\epsilon_{y,x} = \frac{d_y}{d_x}\frac{x}{y} \tag{4}$$

Note 1. For those interested in the derivation of this formula, consider the continuous equation $\frac{\%\Delta y}{\%\Delta x}$. Let's rewrite it so that we get $\frac{\Delta y}{\Delta x}\frac{x}{y}$. Now, if Δx and Δy are both very small, Δx and Δy becomes $\frac{d_y}{d_x}$ and we are left with $\dfrac{d_y}{d_x}\dfrac{x}{y}$.

And there we have it. We were able to derive two very important equations in Economics. I should note here that the specific variables for x and y can be anything. Remember: elasticity is a measure of responsiveness. Some famous examples of elasticity include

1. **Income Elasticity of Demand**: This is a measure of percent change in quantity demanded in response to a 1% change in consumer income.
2. **Price Elasticity of Demand**: This is a measure of percent change in quantity demanded in response to a 1% change in price.
3. **Cross-Price Elasticity of Demand**: This is a measure of percent change in quantity demanded for one good in response to a 1% change in price of another good.

Try to write those elasticity formulas based on the notation we have used hitherto. Solutions will be at the bottom of the page.

1.2 Practice Problems

1.2.1 Basic Problems[1]

1. If demand is represented by $Q = 10 - 2P$, calculate the price elasticity.
2. Elasticity of supply is $\frac{3}{4}$, and a price change causes q to decrease by 9%. What is the percentage price change?
3. Suppose $Q = 10000 - 10P$. P is initially 500 but decreases to 400, what is the price elasticity of demand?
   (a) Now suppose the current market price is \$400. A firm asks you, the economic expert, whether or not they should increase the price. What do you say, and how do you justify your answer?

[1] By "basic" I do not mean trivial. It is fine if you struggle with these at first. These types of problems are typically just calculations and do not test your conceptual understanding.

1.2.2 Medium Problems

1. Demand for good x: $Q_x = 10 - 2P_x + 3P_y$. Calculate the cross-price elasticity.

2. Elasticity of demand for a consumer's income is $\frac{-1}{2}$. A consumer's income is initially \$40,000/year and they buy 50 of good x each year. How much of good x will they buy if income rises to \$56,000 per year?

1.2.3 Hard Problems

1. How does adding a positive constant linear term to $f(x)$ affect elasticity? Assume f is strictly positive and increasing.

2. How does taking the inverse of a function affect its elasticity?

1.3 Solutions

1.3.1 Basic Problems

1. $\epsilon_{Q,P} = \frac{d_Q}{d_P}\frac{P}{Q} = (-2)\frac{P}{Q}$

   $= (-2)\frac{P}{10 - 2P}$

   $= \frac{-2P}{10 - 2P}$

2. $\epsilon_{Q,P} = \frac{\Delta\%Q}{\Delta\%P}$

   $\frac{3}{4} = \frac{-9\%}{\Delta\%P}$

   $\Delta\%P = (-9\%)\frac{4}{3}$

   $\Delta\%P = -12\%$

3. $P_1 = 500, P_2 = 400. Q_1 = 10,000 - 10(500) = 5000. Q_2 = 10,000 = 10(400) = 6000.$

   $\epsilon = \frac{\frac{6000 - 5000}{5000}}{\frac{400 - 500}{500}}$

   $= \frac{.2}{-.2}$

$= 1$

(a) Since it is unit elastic, I would recommend not changing the price at all. At this point, the percentage change in price would equal a percentage change in quantity. If we had an inelastic good, for example, a change in price would be bigger than a change in quantity demanded.

1.3.2 Medium Problems

1. $\epsilon_{P_y} = \dfrac{dQ_x}{dP_y} \text{ x } \dfrac{P_y}{Q_x}$

   $= (3)\dfrac{P_y}{10 - P_x + 3P_y}$

   $= \dfrac{3P_y}{10 - P_x + 3P_y}$

2. $\epsilon_{Q,I} = \dfrac{\%\Delta Q}{\%\Delta P}$

   $= \%\Delta I = \dfrac{56 - 40}{40} = \dfrac{16}{40} = .4$

   $= \%\Delta Q = \epsilon \text{ x } \%\Delta I = \dfrac{-1}{2} \text{ x } \dfrac{2}{5} = \dfrac{-1}{5}$

   $\dfrac{Q_2 - 50}{50} = \dfrac{-1}{5}$

   $Q_2 - 50 = -10$

   $Q_2 = 40$

1.3.3 Hard Problems

1. $\epsilon_{f(x)} = \dfrac{dy}{dx} \text{ x } \dfrac{x}{y}$

   We need to create a new function: $g(x) = f(x) + c$

   $\epsilon_{g(x)} = \dfrac{dy}{dx} \text{ x } \dfrac{x}{y + c}$

   $\therefore \epsilon_{f(x)} > \epsilon_{g(x)}$

2. The inverse of $f(x)$ is $\frac{1}{f(x)}$. Let's solve for a specific function before generalizing our answer.

   $f(x) = x^2$ so the inverse is $f^-1(x) = x^{-2}$.

$\epsilon = \dfrac{df}{dx}\ \dfrac{x}{f} = (2x)\ \dfrac{x}{x^2}$

$\epsilon = \dfrac{2x^2}{x^2} = 2$

Now for the inverse: $\epsilon = \dfrac{df}{dx}\ \dfrac{x}{f} = (-2x^{-3})\ \dfrac{x}{x^{-2}}$

$\epsilon = \dfrac{-\ 2x^{-2}}{x^{-2}} = -2$

We see that taking the inverse of a function keeps the value the same but changes its sign. A general form for this would go as follows:

$\epsilon_{y,x} = \dfrac{dy}{dx}\ \dfrac{x}{y}.\ g = \frac{1}{y}.$

$\epsilon_{g,x} = \dfrac{dg}{dx}\ \dfrac{x}{g}$

$\epsilon_{g,x} = \dfrac{1}{-y^2}\ (g)\ \dfrac{x}{\frac{1}{y}}$

$-g\frac{x}{y} = -\epsilon_{y,x}$